Theorem 4. (Arzela-Ascoli Theorem). Let K be a compact subset of $\mathbb{R}^p$ and let $\mathscr{F}$ be a subset of the set of continuous functions on K, i.e. $\mathscr{F} \subset \{f : K \to \mathbb{R}^q \mid f \text{ is continuous on } K\}$. Then the following properties are equivalent:
(1) The family $\mathscr{F}$ is bounded and uniformly equicontinuous on K.
(2) Every sequence from $\mathscr{F}$ has a subsequence which is uniformly convergent on K $\boxed{\textbf{to a uniformly continuous function } f}$ **(which may not belong to $\mathscr{F}$)**.

Proof of (2) $\Rightarrow$ **$\mathscr{F}$ is bounded :** Suppose that $\mathscr{F}$ is not bounded, then there exists a sequence $\{f_n\}_{n\in\mathbb{N}} \subset \mathscr{F}$ such that $\|f_n\|_K \geq n$, i.e $\exists\, x_n \in K$ such that $\|f_n(x_n)\| \geq n$. This implies that f_n cannot have any subsequence that converges uniformly on K, since if f_n had a subsequence, still denoted by f_n, that converges uniformly on K $\boxed{\textbf{to a uniformly continuous function } f}$ **(which may not belong to $\mathscr{F}$)**, then we have $\infty = \lim\limits_{n\to\infty} \|f_n\|_K \leq \lim\limits_{n\to\infty} \|f_n - f\|_K + \lim\limits_{n\to\infty} \|f\|_K < \infty$ which is a contradiction.

Proof of (2) $\Rightarrow$ **$\mathscr{F}$ is uniform equicontinuous:** Suppose that $\mathscr{F}$ is not uniformly equicontinuous on K, then there exists an $\epsilon_0 > 0$, such that for each $n \in \mathbb{N}$, there exist $f_n \in \mathscr{F}$, and $x_n, y_n \in K$ satisfying that $\|x_n - y_n\| < \dfrac{1}{n}$ and $\|f_n(x_n) - f_n(y_n)\| \geq \epsilon_0$. This implies that f_n cannot have any subsequence that converges uniformly on K, since if f_n had a subsequence, still denoted by f_n, that converges uniformly on K $\boxed{\textbf{to a uniformly continuous function } f}$ **(which may not belong to $\mathscr{F}$)**, thus

$$0 < \epsilon_0 \leq \lim_{n\to\infty} \|f_n(x_n) - f_n(y_n)\| = \lim_{n\to\infty} \|f_n(x_n) - f(x_n) + f(x_n) - f(y_n) + f(y_n) - f_n(y_n)\|$$
$$\leq \lim_{n\to\infty} \|f_n(x_n) - f(x_n)\| + \lim_{n\to\infty} \|f(x_n) - f(y_n)\| + \lim_{n\to\infty} \|f_n(y_n) - f(y_n)\|$$
$$\leq \lim_{n\to\infty} \|f_n - f\|_K + \lim_{n\to\infty} \|f(x_n) - f(y_n)\| + \lim_{n\to\infty} \|f_n - f\|_K = 0,$$

which is a contradiction. Note that, in the last equality,
$\lim\limits_{n\to\infty} \|f_n - f\|_K = 0$ since f_n converges to f uniformly on K, and
$\lim\limits_{n\to\infty} \|f(x_n) - f(y_n)\| = 0$ since f is uniformly continuous on K, and $\|x_n - y_n\| < \dfrac{1}{n}$.

Proof of (1) $\Rightarrow$ (2) **Step** (1)**: Using diagonal process to extract a subsequence from a given sequence $\{f_n\} \subset \mathscr{F}$ (a bounded family):** Let $S = K \cap \mathbb{Q}^p = \{x_i\}_{i\in\mathbb{N}}$. Note that S, called a dense subset of K, is a countable set, and $\overline{S} = K$, i.e. for each $x \in K$, and for each $\delta > 0$, there exists $x_i \in S$ such that $x_i \in B(\delta, x) = \{z \in \mathbb{R}^p \mid \|z - x\| < \delta\}$.
Suppose that $\mathscr{F}$ is bounded and $\{f_n\}$ is any sequence in $\mathscr{F}$, then, since
$(*)$ **$\{f_n(x_1)\}$ is bounded in $\mathbb{R}^q \Rightarrow \{f_n\}$ has a subsequence, denoted $\{f_n^1\}$, converges at x_1.**
Next, since
$(*)$ **$\{f_n^1(x_2)\}$ is bounded in $\mathbb{R}^q \Rightarrow \{f_n^1\}$ has a subsequence, denoted $\{f_n^2\}$, converges at x_2, and x_1 (since $\{f_n^2\}$ is a subsequence of $\{f_n^1\}$ that converges at x_1).**
Inductively, for each $k \geq 2$, since

$$(*)\ \begin{cases} \{f_n^k(x_{k+1})\}\ \textbf{is bounded in}\ \mathbb{R}^q \Rightarrow \{f_n^k\}\ \textbf{has a subsequence, denoted}\ \{f_n^{k+1}\},\ \textbf{converges at} \\ x_{k+1},\ \textbf{and}\ x_j\ (\textbf{since}\ \{f_n^{k+1}\}\ \textbf{is a subsequence of}\ \{f_n^j\}\ \textbf{that converges at}\ x_j\ \textbf{for each}\ j = 1,\ldots,k.) \end{cases}$$

$$
\begin{array}{ccccccccc}
 & & & & \{f_n\} & & & & \\
 & & & & \cup & & & & \\
\{ & f_1^1 & f_2^1 & f_3^1 & \cdots & f_k^1 & \cdots & \}(x_1) & \to f(x_1) \\
 & & & & \cup & & & & \\
\{ & f_1^2 & f_2^2 & f_3^2 & \cdots & f_k^2 & \cdots & \}(x_2) & \to f(x_2) \\
 & & & & \cup & & & & \\
\{ & f_1^3 & f_2^3 & f_3^3 & \cdots & f_k^3 & \cdots & \}(x_3) & \to f(x_3) \\
 & & & & \cup & & & & \\
 & & & & \cdots & & & & \\
 & & & & \cup & & & & \\
\{ & f_1^k & f_2^k & f_3^k & \cdots & f_k^k & \cdots & \}(x_k) & \to f(x_k)
\end{array}
$$

By setting $g_n = f_n^n$, we obtain a subsequence of $\{f_n\}$. Note that for each $k \in \mathbb{N}$, and for each $1 \le j \le k$, $g_n = f_n^n \in \{f_m^j\}_{m\in\mathbb{N}}$ whenever $n \ge j$. This implies that $\lim\limits_{n\to\infty} g_n(x_j) = f(x_j)$ for each $1 \le j \le k$ and for all $k \in \mathbb{N}$. Hence, $\lim\limits_{n\to\infty} g_n(x_j) = f(x_j)$ for each $x_j \in S$.

Proof of (1) $\Rightarrow$ (2) **Step** (2)**: Using the uniform equicontinuity of $\mathscr{F}$ to show that the subsequence g_n (of f_n) converges uniformly to f on K:** In particular, we shall show that the sequence g_n (which is a subsequence of $\{f_n\} \subset \mathscr{F}$) satisfies the **Cauchy criterion** for uniform convergence on K $\boxed{\textbf{to a uniformly continuous function } f}$ **(which may not belong to $\mathscr{F}$)**, i.e. for each $\epsilon > 0$ and for each $x \in K$, there exists $L \in \mathbb{N}$ such that if $m, n \ge L$, then $\|g_n(x) - g_m(x)\| < \epsilon$.

For each $\epsilon > 0$, since $\mathscr{F}$ is uniformly equicontinuous on K, there exists a $\delta = \delta(\epsilon) > 0$ such that if $x, y \in K$ satisfying that $\|x - y\| < \delta$, then $\|h(x) - h(y)\| < \epsilon$ for all $h \in \mathscr{F}$. This implies that **for each each** $x \in K$, since $\overline{S} = K \subset \cup_{i=1}^{\infty} B(\delta, x_i)$, and each $g_n \in \mathscr{F}$, there exists an $x_i \in S$ such that

$(*)$ $\|x - x_i\| < \delta \Longrightarrow \|g_n(x) - g_n(x_i)\| < \dfrac{\epsilon}{3}$ for all $n \in \mathbb{N}$.

Also, for the same $\epsilon > 0$, since $\lim\limits_{n\to\infty} g_n(x_i) = f(x_i)$, **there exists $L \in \mathbb{N}$ such that**

$(\dagger)$ for any $m, n \ge L$, we have $\|g_m(x_i) - g_n(x_i)\| < \dfrac{\epsilon}{3}$.

Therefore, combining $(*)$ and $(\dagger)$, we have for each $x \in K$ and **for any** $m, n \ge L$, **we have**

$\|\mathbf{g_n}(\mathbf{x}) - \mathbf{g_m}(\mathbf{x})\| \le \|g_n(x) - g_n(x_i)\| + \|g_n(x_i) - g_m(x_i)\| + \|g_m(x_i) - g_m(x)\| < \dfrac{\epsilon}{3} + \dfrac{\epsilon}{3} + \dfrac{\epsilon}{3} = \epsilon.$

This implies that for any $m, n \ge L$, $\|g_n - g_m\|_K < \epsilon$, and, g_n **converges uniformly on** K $\boxed{\textbf{to a uniformly continuous function } f}$ **(which may not belong to $\mathscr{F}$).**

Study guide of Chapter 3:
Outlines of (3.1): Apply the Implicit Function Theorem to study the following question:

Q: Let W be an open subset of $\mathbb{R}^m \times \mathbb{R}^n$, $F : W \subset \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$ be a (vector-valued) function of class C^1, and (a, b) be a point in $W \subset \mathbb{R}^m \times \mathbb{R}^n$ satisfying $F(a, b) = 0$. When is there
(1) a function $f(x)$, defined in some open set in $\mathbb{R}^n$ containing a, and
(2) an open set $U \subset W \subset \mathbb{R}^{m+n}$ containing (a, b),
such that for $(x, y) \in U$,

$$F(x, y) = 0 \Longleftrightarrow y = f(x)?$$

Read e.g. Theorem 3.1, Theorem 3.9 in the book, and Exercises $1-3, 5-7, 9$ in (3.1)

Outlines of (3.2), (3.3), (3.5): There are three common ways of **locally representing smooth** $k-$**dimensional manifolds** (e.g. curves, or surfaces if $k = 1$ or 2.) in $\mathbb{R}^n$:

Type (1) : as the graph of a function, $y = f(x)$, where f is of class C^1, $x \in B$, and B is a connected open set in $\mathbb{R}^k$;
Type (2) : as the locus of an equation $F(x, y) = 0$, where F is of class C^1;
Type (3) : parametrically, as the range of a C^1 function $g : U \subset \mathbb{R}^m \to \mathbb{R}^n$ whose differential Dg has rank k everywhere in a connected open set U.

Note that **if a manifold can be represented locally in Type** (1), **then it can be represented in both Types** (2) **and** (3) by setting $F(x, y) = y - f(x)$, and $g(x) = (x, f(x))$ for $x \in B \subset \mathbb{R}^k$, respectively. It is natural to ask the following converse, i.e.

Q: When can a manifold given locally in Types (2) or (3) be represented in Type (1)?

Read e.g. Theorem 3.11, Theorem 3.15, Theorem 3.21 in the book, and Exercises $1-3$ in (3.2), and $1, 2$ in (3.3).

Outlines of (3.4), (3.5): A question concerned here is:

Q: When can a mapping (or a transformation) $f : \mathbb{R}^n \to \mathbb{R}^n$ of class C^1 have a local inverse? How smooth is the inverse (if it exists)?

Read e.g. the following The Inverse Function Theorem in the book.
Theorem 3.18: Let U and V be open sets in $\mathbb{R}^n$, $a \in U$, and $b = f(a)$. Suppose that $f : U \to V$ is a mapping of class C^1 and the Fréchet derivative $Df(a)$ is invertible (i.e. the Jacobian det $Df(a)$ is nonzero). Then $\exists$ neighborhoods $M \subset U$ and $N \subset V$ of a and b, respectively, so that f is a one-to-one map from M onto N, and the inverse map f^{-1} from N to M is also C^1. Moreover, if $y = f(x) \in N$, $D(f^{-1})(y) = [Df(x)]^{-1}$.

Note that if f is C^1 with nonzero det $Df(a)$ at an **interior** point a, then $b = f(a)$ is also an interior point of $f(U)$, i.e. **The Inverse Function Theorem also tells us that when an interior point in the domain is mapped to an interior point in the range.**

Also, read e.g. Theorem 3.20, Theorem 3.22 in the book, and Exercises $1, 2, 5$ in (3.4) and 1 in (3.5).